Bayesian Nonparametric Adaptive Control Using Gaussian Processes Girish Chowdhary, Hassan A. Kingravi, Jonathan P. How, Senior Member, IEEE, and Patricio A. Vela, Member, IEEE [1]

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***Abstract*— This paper presents a novel Model Reference Adaptive Control framework utilizing Gaussian Processes. Previous work focused on a Radial Basis Function Network approach which was not robust to model dynamics and which relied on pre-initialized centers for the Radial Basis Functions within the domain of operation. The Gaussian Process approach adapts to stochastic dynamics and adjust with model, without relying on pre-initialized information. This architecture was demonstrated on the Wing-Rock problem and shown to work successfully even when the system is driven outside the normal domain of operation.**

I. INTRODUCTION

Model Reference Adaptive control is useful in a wide variety of situations where there is significant uncertainty in the system model. Previous approaches utilized Radial Basis Functions, which need to be pre-allocated with centers within the domain of operation, and are not robust to large changes in operating conditions. This paper presents a novel architecture which utilizes Gaussian Processes for Model Reference Adaptive Control. In this approach, the uncertainty is encoded in the Gaussian process model and the architecture adjusts to changes in system dynamics without relying on pre-allocated information. This paper provides bounds on stability for this architecture as well as demonstrates its validity and improvement over the Radial Basis Function approach for the Wing Rock Problem.

II. APPROXIMATE MODEL INVERSION BASED MRAC

First consider the system described below:

$$\dot{x}_1\left(t\right) = x_2\left(t\right) \tag{1}$$

$$\dot{x}_2\left(t\right) = f\left(x\left(t\right)\right) + b\left(x\left(t\right)\right)\delta\left(t\right) \tag{2}$$

We define a model in terms of the psuedoinput, υ, and the approximate inversion model, $\hat{f}\left(x\right) + \hat{b}\left(x\right)\delta\left(t\right)$. The transformed system is shown below.

$$\dot{x}_1\left(t\right) = x_2\left(t\right) \tag{3}$$

$$\dot{x}_2\left(t\right) = \upsilon + \left(f - \hat{f}\right) + \left(b - \hat{b}\right)\delta\left(t\right) \tag{4}$$

*This work was done towards completing the project component of ECE555

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The input δ is then defined terms of the parameters just mentioned.

$$\delta\left(t\right) = \hat{b}^{-1}\left(x\right)\left(\upsilon - \hat{f}\left(x\right)\right) \tag{5}$$

The model is then rewritten as.

$$\dot{x}_1\left(t\right) = x_2\left(t\right) \tag{6}$$

$$\dot{x}_2\left(t\right) = \upsilon + \left(f - \hat{f}\right) + \left(b - \hat{b}\right)\hat{b}^{-1}\left(\upsilon - \hat{f}\left(x\right)\right) \tag{7}$$

It is trivial to see this will recover the old model as follows.

$$\dot{x}_1\left(t\right) = x_2\left(t\right) \tag{8}$$

$$\dot{x}_2\left(t\right) = f\left(x\right) + b\left(x\right)\cdot\delta\left(t\right) \tag{9}$$

We will write the system in terms of an updated state z, which includes the state space as well as our input δ.

$$z = \begin{bmatrix} x_1^T & x_2^T & \delta^T \end{bmatrix} \tag{10}$$

We define the function Δ as follows.

$$\Delta\left(z\right) = \left(f - \hat{f}\right) + \left(b - \hat{b}\right)\hat{b}^{-1}\left(\upsilon - \hat{f}\left(x\right)\right) \tag{11}$$

The system can then be rewritten in terms of this parameter.

$$\dot{x}_2\left(t\right) = \upsilon\left(z\right) + \Delta\left(z\right) \tag{12}$$

The updated matrix form is as below.

$$\left\{ \begin{array}{c} \dot{x}_1\left(t\right) = x_2\left(t\right) \\ \dot{x}_2\left(t\right) = \upsilon\left(z\right) + \Delta\left(z\right) \end{array} \right\} \tag{13}$$

We now write the whole system in matrix form.

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot \left[\upsilon + \Delta\right] \tag{14}$$

The psuedoinput is defined below.

$$\upsilon = \upsilon_{pd} + \upsilon_{rm} - \upsilon_{ad} \tag{15}$$

This includes a proportional differential control, as well as a reference model and an adaptive term. We define the proportional differential control in terms of the error.

$$\upsilon_{pd} = \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} \cdot e \tag{16}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1,rm} \\ x_2 - x_{2,rm} \end{bmatrix} \tag{17}$$

The psuedoinput then becomes.

$$\upsilon = \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} e + \upsilon_{rm} - \upsilon_{ad} \tag{18}$$

The psuedoinput for the reference model will be made use of below.

$$\upsilon = \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} e + \dot{x}_{rm} - \upsilon_{ad}, \upsilon_{rm} = \dot{x}_{rm} \tag{19}$$

We desire an expression for the error. We start with the system below.

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 - \dot{x}_{1,rm} \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_{1,rm} \\ x_2 - x_{2,rm} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot [\upsilon + \Delta] \end{aligned} \tag{20}$$

Using the expression for the pseudoinput this expands

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 - \dot{x}_{1,rm} \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_{1,rm} \\ x_2 - x_{2,rm} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot [-K_1 e_1 - K_2 e_2 + \upsilon_{rm} - \upsilon_{ad} + \Delta] \end{aligned} \tag{21}$$

We then separate out the psuedoinput for the reference model.

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 - \dot{x}_{1,rm} \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_{1,rm} \\ x_2 - x_{2,rm} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot [-K_1 e_1 - K_2 e_2 - \upsilon_{ad} + \Delta] + \begin{bmatrix} 0 \\ \dot{x}_{2,rm} \end{bmatrix} \end{aligned} \tag{22}$$

Moving this to the other side yields the desired relation for the error.

$$\begin{aligned} \dot{e} = \begin{bmatrix} \dot{x}_1 - \dot{x}_{1,rm} \\ \dot{x}_2 - \dot{x}_{2,rm} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \cdot e \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot [-K_1 e_1 - K_2 e_2 - \upsilon_{ad} + \Delta] \end{aligned} \tag{23}$$

We simplify this below.

$$\dot{e} = \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix} \cdot e + \begin{bmatrix} 0 \\ I \end{bmatrix} \cdot [\Delta - \upsilon_{ad}] \tag{24}$$

We then simplify the error equation in terms of the matrices A and B.

$$\dot{e} = A \cdot e + B \cdot [\Delta - \upsilon_{ad}] \tag{25}$$

$$A = \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{26}$$

We will now show that the system is stable with this architecture. Consider the candidate Lyupanov function V.

$$V = \frac{1}{2} e^T P e \tag{27}$$

The time differential of this function is a known result.

$$\dot{V} = \frac{1}{2} \left[\dot{e}^T P e + e^T P \dot{e} \right] \tag{28}$$

Utilizing the expression for the time derivative of the error.

$$\begin{aligned} \dot{V} &= \tfrac{1}{2} \left(Ae + B \left(\Delta - \upsilon_{ad} \right) \right)^T Pe \\ &+ \tfrac{1}{2} e^T P \left(Ae + B \left(\Delta - \upsilon_{ad} \right) \right) \end{aligned} \tag{29}$$

We separate the terms dependent on A and B. We choose K_1 and K_2 such that A is Hurwitz and the system is stable.

$$\begin{aligned} \dot{V} &= \tfrac{1}{2} \left((Ae)^T Pe + e^T P (Ae) \right) \\ &+ \tfrac{1}{2} \left(\left(B \left(\Delta - \upsilon_{ad} \right) \right)^T Pe + e^T P \left(B \left(\Delta - \upsilon_{ad} \right) \right) \right) \end{aligned} \tag{30}$$

We evaluate the transpose

$$\begin{aligned} \dot{V} &= \tfrac{1}{2} \left(e^T A^T Pe + e^T PAe \right) \\ &+ \tfrac{1}{2} \left(B^T Pe + e^T PB \right) \left(\Delta - \upsilon_{ad} \right) \end{aligned} \tag{31}$$

Because A is Hurwitz, we use the Lyupanov Equation along with the fact that $B^T Pe$ is scalar.

$$\begin{aligned} \dot{V} &= \tfrac{1}{2} e^T \left(A^T P + PA \right) e \\ &+ \tfrac{1}{2} \left(B^T Pe + \left(B^T Pe \right)^T \right) \left(\Delta - \upsilon_{ad} \right) \end{aligned} \tag{32}$$

The final expression for the time derivative is shown below.

$$\dot{V} = -\frac{1}{2} e^T Q e + \left(B^T P e \right) \left(\Delta - \upsilon_{ad} \right) \tag{33}$$

If we find an expression for Δ then we arrive at the result.

$$\dot{V} = -\frac{1}{2} e^T Q e, \Delta = \upsilon_{ad} \tag{34}$$

III. Adaptive Control Using GP Regression

In the previous section, we successfully determined a model reference adaptive control architecture to ensure system stability while tracking the desired reference signal. However, we did not account of the stochastic nature of the model itself. This section details how a Gaussian Process framework is used to capture the uncertainty in the model and the to calculate the best estimate of the psuedoinput for the observed system states.

We model the uncertainty, Δ, as a Gaussian Process.

$$\Delta(z) \sim GP \left(m(z), k(z, z') \right) \tag{35}$$

In this case, $k(z, z')$, is a real positive definite kernel function, and where the mean is assumed to lie in the kernel space. This space of functions is a Hilbert space, and is defined such that all functions within the space have a finite norm written in terms of kernel basis functions.

$$g \in H : \|g\|_H^2 = \sum_i \sum_j \left\langle \alpha_i \psi(z_i), \alpha_j \psi(z_j) \right\rangle \tag{36}$$

Using the fact that the kernel is defined as an inner product of basis functions.

$$\|g\|_H^2 = \sum_i \sum_j \alpha_i \alpha_j \left\langle \psi(z_i), \psi(z_j) \right\rangle \tag{37}$$

$$\|g\|_H^2 = \sum_i \sum_j \alpha_i \alpha_j k(z_i, z_j) \tag{38}$$

We see that the above expression is indeed finite.

A. GP Regression and Reproducing Kernel Hilbert Spaces

Let Z be the set of state measurements.

$$Z_\tau = \{z_1, ..., z_\tau\} \tag{39}$$

This set defines a covariance matrix which is written in terms of the kernel functions as below.

$$K_{ij} = k\left(z_i, z_j\right) \tag{40}$$

For each state measurement, the observed output is given by the mean for each sample plus a Gaussian noise term ϵ.

$$y_{z_i} = m\left(z_i\right) + \varepsilon_i, \varepsilon_i \sim N\left(0, \omega^2\right) \tag{41}$$

Therefore, the state measurements a transformed to observed system outputs, and our desire is to extract the state data from this measurement.

$$Z_\tau = \{z_1, ..., z_\tau\} \to y = \{y_1, ..., y_\tau\}^T \tag{42}$$

We choose a Gaussian distribution for our kernel functions, as these are valid basis which map into an infinite dimensional space.

$$k\left(z, z'\right) = e^{\left(-\frac{\left\|z-z'\right\|}{2\mu^2}\right)} \tag{43}$$

We know that each kernel function can be written in terms of inner products of bases in the kernel space.

$$k\left(z_i, z_j\right) = \left\langle \psi\left(z_i\right), \psi\left(z_j\right)\right\rangle \tag{44}$$

This allows us to wright the desired measurement $m(z)$ in terms of the bases vectors weighted by a vector β. We rewrite this infinite dimensional product in terms of a sum of inner products in the kernel space, which are of finite dimensional.

$$m\left(z\right) = \beta^T \psi\left(z\right) = \sum_{i=1}^{n} \beta_j \left\langle \psi\left(z_i\right), \psi\left(z\right)\right\rangle \tag{45}$$

Each new data point $z_{\tau+1}$ will lead to a resultant update in the posterior Gaussian distribution for the data. Each new observation $y_{\tau+1}$ is jointly distributed with the previous observation sequence y^τ.

$$\begin{bmatrix} y_\tau \\ y_{\tau+1} \end{bmatrix} \sim N\left(0, \begin{bmatrix} K\left(Z_\tau, Z_\tau\right) + \omega^2 I & k_{z_{\tau+1}} \\ k_{z_{\tau+1}}^T & k_{z_{\tau+1}}^* \end{bmatrix}\right) \tag{46}$$

This distribution is written in terms of K, the covariance matrix up to time τ, in terms of $k_{z_{\tau+1}}$, the joint covariance of the new measurement with that of the previous sequence, and in terms of, $k_{z_{\tau+1}}^*$, the covariance of the new measurement.

$$k_{z_{\tau+1}} = K\left(Z_\tau, Z_\tau\right) \tag{47}$$

$$k_{z_{\tau+1}}^* = k\left(z_{\tau+1}, z_{\tau+1}\right) \tag{48}$$

It is known that the conditional distribution of the latest observation given the previous observation sequence is also Gaussian.

$$P\left(y_{\tau+1} \middle| Z_\tau, y_\tau, z_{\tau+1}\right) \sim N\left(\hat{m}_{\tau+1}, \hat{\Sigma}_{\tau+1}\right) \tag{49}$$

This distribution is written in terms of the best estimate $\hat{m}_{\tau+1}$ and the error covariance $\hat{\Sigma}_{\tau+1}$.

$$\hat{m}_{\tau+1} = \beta_{\tau+1}^T k_{z_{\tau+1}} \tag{50}$$

$$\hat{\Sigma}_{\tau+1} = k_{z_{\tau+1}}^* - k_{z_{\tau+1}}^T C_\tau k_{z_{\tau+1}} \tag{51}$$

These in tern involve the inverse covariance of the observation sequence, C_τ, and the product of this term with the new observation, $\beta_{\tau+1}$.

$$C_\tau = \left(K\left(Z_\tau, Z_\tau\right) + \omega^2 I\right)^{-1} \tag{52}$$

$$\beta_{\tau+1} = C_\tau y_\tau \tag{53}$$

As one can observe from the constants above, the inverse must be computed in this scenario. Because computing the inverse is costly, we will try to expedite computation by pruning the space of basis functions while maintaining an effective state estimate given all the current data.

B. GP Bayesian Nonparametric Model-Based MRAC

We now discuss how the psuedoinput is computed based on the data. Recall that the adaptive psuedoinput is the only thing that is required to drive the error to zero, given the model discussed in the previous section. We desire an adaptive psuedoinput which is modeled by a Gaussian process.

$$\upsilon_{ad}\left(z\right) \sim GP\left(\hat{m}\left(z\right), k\left(z, z'\right)\right) \tag{54}$$

We define the coefficient α_τ to be the stochastic projection of the new data onto the old within the kernel space, as given by the known best estimate for the Gaussian process.

$$\alpha_\tau = K_{Z_\tau}^{-1} k_{z_{\tau+1}} \tag{55}$$

We define the length of the residual $\gamma_{\tau+1}$, to be the distance between the basis of the new data point, and the sum of the projections onto the other bases functions as given by the product of the coefficients α with each basis.

$$\gamma_{\tau+1} = \min_{\alpha_i} \left\| \sum_{i=1}^{\tau} \alpha_i \psi\left(z_i\right) - \psi\left(z_{\tau+1}\right) \right\|^2 \tag{56}$$

Matching the components of this expression with the parameters in the Gaussian process defined above, we arrive at an expression for $\gamma_{\tau+1}$ which includes the known parameters.

$$\gamma_{\tau+1} = k_{\tau+1}^* - k_{z_{\tau+1}}^T \alpha_\tau \tag{57}$$

Our algorithm will then prune the basis functions such that those with the maximum distance from the previous bases will be chosen, ensuring that the basis functions span the space as well as possible. The algorithm which realizes this is Csato's Algorithm [2], an algorithm is now popular for the realization of Gaussian Process Estimators.

IV. Analysis of Stability

A. Stochastic Stability Theory for Switched Systems

We begin by considering the stochastic differential equation below, which represents the state in terms of a perturbation from the reference model actuated via a Weiner process $\zeta(t)$.

$$dx(t) = F\left(t, x(t)\right)dt + H_\sigma\left(t, x(t)\right)d\zeta(t) \tag{58}$$

The dimensionality of this equation is detailed below.

$$x \in R^{n_s}, \zeta(t) \in R^{n_2}, \sigma(t) \in N \tag{59}$$

$$F(t,x) \in R^{n_s}, H_\sigma(t,x) \in R^{n_s \times n_2} \tag{60}$$

Finally, we initialize the process as follows.

$$x(0) = 0, F(t,0) = 0, H_\sigma(t,0) = 0 \tag{61}$$

For the following analysis, we assume that the functions F and H_σ satisfy the following Lipschitz criteria.

$$\begin{aligned}&\left\|F(t,x) - F(t,y)\right\| + \left\|H_\sigma(t,x) - H_\sigma(t,y)\right\| \\ &\leq B\left\|x-y\right\|\end{aligned} \tag{62}$$

The following definitions introduce the stability bounds which we desire to prove our Model Reference Adaptive Control Architecture will realize.
Definition 1: The process $x(t)$ is mean square ultimately bounded uniformly in σ if:

$$\exists k \in R : \forall(t, \sigma, x_o), \lim_{t\to\infty} E_{x_o}\left[\left\|x(t)\right\|^2\right] \leq k \tag{63}$$

Definition 2: A process $x(t)$ is exponentially means square ultimately bounded uniformly in σ if:

$$\begin{aligned}&\exists(k, c, \alpha) : \forall(t, \sigma, x_o), \\ &\lim_{t\to\infty} E_{x_o}\left\|x(t)\right\|^2 \leq k + c\|x_o\|^2 e^{-\alpha}\end{aligned} \tag{64}$$

Keeping in mind the above definitions, we will analyze the stability of the system governed by the above stochastic differential equation. We will define a Lyupanov function in terms of the error and a positive definite matrix P. This will ensure system stability if A is Hurwitz.

$$V(t,x) = \frac{1}{2}e^T Pe, P > 0 \tag{65}$$

First, we compute the differential generator for the function $V(t,x)$ is:

$$\begin{aligned}&\tfrac{d\left[V(t,x)\right]}{dt} = \tfrac{\partial\left[V(t,x)\right]}{\partial t} + \sum_j F_j(t,x)\tfrac{\partial\left[V(t,x)\right]}{\partial x_j} \\ &+\tfrac{1}{2}\sum_{i,j}\left[H_\sigma H_\sigma^T\right]_{ij}(t,x)\tfrac{\partial^2\left[V(t,x)\right]}{\partial x_j \partial x_i}\end{aligned} \tag{66}$$

The following lemma is taken as a corollary to that presented in [3].
Lemma 1: Let $x(t)$ satisfy the above stochastic differential equation and $V(t,x)$ be the class of function which are twice continuously differentiable with respect to x and once continuously differentiable with respect to t. If:

$$\exists(k_1, k_2) : \frac{dE\left[V(t,x)\right]}{dt} \leq k_1 - k_2 V(t,x) \tag{67}$$

Then,

$$\begin{aligned}&\forall t \geq 0, \\ &E_{x_o}V(t,x) \leq V(0,x_o)e^{-k_2 t} + \tfrac{|k_1|}{k_2}\left(1 - e^{-k_2 t}\right)\end{aligned} \tag{68}$$

The following theorem proves ultimate boundedness for the stochastic differential equation described above.
Theorem 1: Let x(t) be a solution to the above stochastic differential equation, and $V(t,x)$ be the class of function which are twice continuously differentiable with respect to x and once continuously differentiable with respect to t. If:
Assumption 1:

$$\forall\left(\alpha_1 \in R, c_1 \in R^+\right), -\alpha_1 + c_1\|x\|^2 \leq V(t,x) \tag{69}$$

Assumption 2:

$$\forall(\beta_\sigma, c_2), \frac{dEV(t,x)}{dt} \leq \beta_\sigma - c_2 V(t,x) \tag{70}$$

Proof: With $k_1 = \beta_\sigma$ and $k_2 = c_2$, Lemma 1 yields:

$$\begin{aligned}&E_{x_o}\left[V\left(t, x(t)\right)\right] \leq \\ &V(0,x_o)e^{-c_2 t} + \tfrac{|\beta_\sigma|}{c_2}\left(1 - e^{-c_2 t}\right)\end{aligned} \tag{71}$$

Taking the limit of both sides yields a bound for the differential generator.

$$\begin{aligned}&\lim_{t\to\infty} E_{x_o}V\left(t, x(t)\right) \\ &\leq \lim_{t\to\infty}\left[V(0,x_o)e^{-c_2 t} + \tfrac{|\beta_\sigma|}{c_2}\left(1 - e^{-c_2 t}\right)\right]\end{aligned} \tag{72}$$

$$\lim_{t\to\infty} E_{x_o}\left[V\left(t, x(t)\right)\right] \leq \frac{|\beta_\sigma|}{c_2} \tag{73}$$

By Assumption 1:

$$-\alpha_1 + c_1\|x\|^2 \leq V(t,x) \tag{74}$$

Therefore,

$$-\alpha_1 + c_1\|x\|^2 \leq V(t,x) \Rightarrow \|x\|^2 \leq \frac{V(t,x)}{c_1} + \frac{\alpha_1}{c_1} \tag{75}$$

This implies,

$$E_{x_o}\left[\|x\|^2\right] \leq \frac{E_{x_o}\left[V(t,x)\right]}{c_1} + \frac{\alpha_1}{c_1} \tag{76}$$

Taking the limit of both sides and simplifying yields a bound on the expectation of x.

$$\lim_{t\to\infty}\left[E_{x_o}\left[\|x\|^2\right]\right] \leq \lim_{t\to\infty}\left[\frac{E_{x_o}\left[V(t,x)\right]}{c_1} + \frac{\alpha_1}{c_1}\right] \tag{77}$$

$$\lim_{t\to\infty}\left[E_{x_o}\left[\|x\|^2\right]\right] \leq \frac{|\beta_\sigma|}{c_2} + \frac{\alpha_1}{c_1} \tag{78}$$

This shows that the process is mean squared ultimately bounded uniformly in σ.

$$\lim_{t\to\infty}\left[E_{x_o}\left[\|x\|^2\right]\right]\leq K, K=\max_{\sigma}\left(\frac{|\beta_\sigma|}{c_2}+\frac{\alpha_1}{c_1}\right) \quad (79)$$

If in addition.
Assumption 3:

$$V(0,x)\leq c_3\|x_o\|^2+\alpha_2 \quad (80)$$

Since,

$$\begin{aligned}&E_{x_o}\left[V\left(t,x(t)\right)\right]\\&\leq V(0,x_o)e^{-c_2t}+\tfrac{|\beta_\sigma|}{c_2}\left(1-e^{-c_2t}\right)\end{aligned} \quad (81)$$

This implies,

$$\begin{aligned}&E_{x_o}\left[V\left(t,x(t)\right)\right]\\&\leq\left(c_3\|x_o\|^2+\alpha_2\right)e^{-c_2t}+\tfrac{|\beta_\sigma|}{c_2}\left(1-e^{-c_2t}\right)\end{aligned} \quad (82)$$

Finally, since.

$$E_{x_o}\left[\|x\|^2\right]\leq\frac{E_{x_o}\left[V(t,x)\right]}{c_1}+\frac{\alpha_1}{c_1} \quad (83)$$

We can change this bound to,

$$\begin{aligned}&E_{x_o}\left[\|x\|^2\right]\\&\left(\tfrac{c_3}{c_1}\|x_o\|^2+\tfrac{\alpha_2}{c_1}\right)e^{-c_2t}+\tfrac{|\beta_\sigma|}{c_2}\left(1-e^{-c_2t}\right)+\tfrac{\alpha_1}{c_1}\end{aligned} \quad (84)$$

Since the second term is monotonically increasing, the previous bound shows that the system is exponentially means square uniformly bounded in σ.

$$E_{x_o}\left[\|x\|^2\right]\leq\left(\frac{c_3}{c_1}\|x_o\|^2+\frac{\alpha_2}{c_1}\right)e^{-c_2t}+K \quad (85)$$

$$K=\frac{|\beta^\sigma|}{c_2}+\frac{\alpha_1}{c_1} \quad (86)$$

■

B. Analysis of Stability for Stochastic System

The uncertain model is described here as a perturbation from the reference model state via a Weiner process.

$$\Delta(z)\sim m\left(z(t)\right)+G_\sigma\left(t,z(t)\right)d\xi \quad (87)$$

The adaptive psuedoinput is described via a Gaussian process based on state data. The best psuedoinput is the estimator given by the mean of this process after it has been updated with all the current data.

$$v_{ad}(z)\sim GP\left(\hat{m}^\sigma(z),k\left(z,z'\right)\right)\Rightarrow\hat{v}_{ad}(z)=\hat{m}^\sigma(z) \quad (88)$$

We consider the familiar differential equation governing the error from the previous section Approximate Model Based Inversed Based MRAC.

$$\dot{e}=Ae+B\left[v_{ad}(z)-\Delta(z)\right] \quad (89)$$

This is rewritten here as a stochastic differential equation for the error involving the uncertainty in the state and the current estimate for the pseudoinput.

$$\begin{aligned}&de=A\cdot e\cdot dt\\&+B\cdot\left(\left(\hat{m}^\sigma(z)-m\left(z(t)\right)\right)\cdot dt+G_\sigma\left(t,z(t)\right)\cdot d\xi\right)\end{aligned} \quad (90)$$

We define an operator representing the difference between the state and the estimate.

$$\varepsilon_m^\sigma(z)=\left(\hat{m}^\sigma(z)-m\left(z(t)\right)\right) \quad (91)$$

The stochastic differential equation for the error is then shown in simplified form below.

$$de=Aedt+B\left[\varepsilon_m^\sigma(z)\,dt+G_\sigma d\xi\right] \quad (92)$$

Now, we will provide a bound on the model error given by $\left\|\Delta(z)-\hat{m}^\sigma(z)\right\|$. We start with the following theorem which is given in [4].
Theorem 2: Define the pseudometric:

$$d_G(t,s)=\sqrt{E\left[\left|G_\sigma d\xi(t)-G_\sigma d\xi(s)\right|^2\right]} \quad (93)$$

Let $N(T,d,\upsilon)$ be the υ-covering number of the space, then the υ-entropy of the space (T,d) is given by:

$$H(T,d,\upsilon)=\log N(T,d,\upsilon) \quad (94)$$

Let $D(t)$ by the diameter of the space T with respect to the metric d_G. Then,

$$E\sup_{t\in T}G_\sigma d\xi(s)\leq C\int_0^{D(T)}H^{\frac{1}{2}}(T,d,\upsilon)\,d\upsilon \quad (95)$$

is a valid bound for the space.
We now prove a corollary which bounds the above metric in a manner more useful for our purposes.
Corollary 3: Let the covariance kernel $k: D_x\times D_x\to R$ of the zero mean Gaussian Process $\left\{G_\sigma d\xi(s)\right\}_{t\in T}$ be Gaussian. The if $D_x\subset R^d$ is a compact set, then:

$$\exists c'\in R^+:\left\|G_\sigma d\xi(s)\right\|\overset{a.s.}{\leq}c' \quad (96)$$

Proof: Proof: The kernel operator is defined as follows.

$$k\left(t,t'\right)=E\left[G_\sigma d\xi(t)\,G_\sigma d\xi\left(t'\right)\right] \quad (97)$$

Therefore, the kernel operator can be used to rewrite the pseudometric as below.

$$\begin{aligned}&d_G(t,s)=\\&\sqrt{E\left[G_\sigma d\xi(t)\,G_\sigma d\xi(t)+G_\sigma d\xi(s)\,G_\sigma d\xi(s)\right]}\end{aligned} \quad (98)$$

$$d_G(t,s)=\sqrt{k(t,t)+k(s,s)-2k(t,s)} \quad (99)$$

Since the kernel operator is axially symmetric, meaning:

$$(t,t)=k(s,s)=\kappa \quad (100)$$

The pseudometric can be simplified again.

$$d_G(t,s)=\sqrt{2\kappa-2k(t,s)}=\sqrt{2}\sqrt{\kappa-k(t,s)} \quad (101)$$

Since the kernel operator is upper bounded by the autocorrelation.

$$\forall\left(t,t'\right)\in T,k\left(t,t'\right)\leq\kappa \tag{102}$$

An upper bound for the pseudometric can be determined.

$$d_G\left(t,s\right)\leq\sqrt{2\kappa} \tag{103}$$

This implies that $D(T)=\sqrt{2\kappa}$ and the integral in Theorem 2 is always finite for this scenario. ■

Given all the previous results, we can now prove the boundedness of the tracking error which we desire. We start with the following theorem.

Theorem 4: Global Approximation Theorem: Let $m^\sigma\left(z\right)$ and $\hat{m}^\sigma\left(z\right)$ represent the state and state estimate as above. Further let the infinity norm of y be:

$$\left\|y\right\|_\infty\leq M^\sigma \tag{104}$$

Then,

$$\left\|\varepsilon_m^\sigma\left(z\right)\right\|\leq\frac{2\kappa^2M^\sigma\sqrt{k_{\max}}}{\omega^4}+\frac{\kappa k_{\max}M^\sigma}{\omega^2} \tag{105}$$

Where the maximum value of the kernel is defined in terms of the centers for the estimation process c_{ϑ_i}.

$$k_{\max}=\max_i\left\|\psi\left(z_i\right)-\psi\left(c_{\vartheta_i}\right)\right\|_H,c_{\vartheta_i}\in D_x \tag{106}$$

Proof: : Let the kernel matrix associated with m^σ be $k_{ij}=\tau^{-2}k\left(z_i,z_j\right)$ and let the kernel matrix associated with the bases for estimation be $\tilde{k}_{ij}=\tau^{-2}k\left(c_{\upsilon(i)},c_{\upsilon(j)}\right)$.
We will use the following properties during this derivation.

$$\left\{\begin{array}{cc}A^{-1}+B^{-1}=B^{-1}\left(A+B\right)A^{-1} & (a)\\ \left\|AB\right\|\leq\left\|A\right\|\left\|B\right\| & (b)\\ \left\|Av\right\|\leq\left\|A\right\|\left\|v\right\| & (c)\\ \left\|A+B\right\|\leq\left\|A\right\|+\left\|B\right\| & (d)\\ \left\|A^{-1}\right\|=\frac{1}{\left\|A\right\|} & (e)\\ \left\|A\right\|=\left\|A^T\right\| & (f)\\ A=P\Lambda P^{-1}\rightarrow\lambda_{\min}\leq\left\|A\right\|\leq\lambda_{\max} & (g)\\ \lambda_{\min}\left(A+B\right)\geq\lambda_{\min}\left(A\right)+\lambda_{\min}\left(B\right) & (h)\end{array}\right\} \tag{107}$$

We define the covariance matrices as below.

$$\left\{\begin{array}{c}K_{ij}=k\left(z_i,z_j\right)\\ \tilde{K}_{ij}=k\left(c_{\upsilon_i},c_{\upsilon_j}\right)\end{array}\right\} \tag{108}$$

Since ϵ is defined as the difference between the mean and the estimate, we start by writing these parameters in terms of the kernel functions.

$$m\left(z\right)=\frac{1}{\tau}\sum_{i=1}^{\tau}\alpha_ik\left(z_i,z\right)=\alpha^Tk_z \tag{109}$$

$$\hat{m}^\sigma\left(z\right)=\frac{1}{\tau}\sum_{i=1}^{\tau}\tilde{\alpha}_ik\left(c_{\upsilon_i},z\right)=\tilde{\alpha}^T\tilde{k}_z \tag{110}$$

The norm of ϵ is then the norm of the difference.

$$\left|m\left(z\right)-\hat{m}^\sigma\left(z\right)\right|=\left\|\alpha^Tk_z-\tilde{\alpha}^T\tilde{k}_z\right\| \tag{111}$$

We now add and subtract $\tilde{\alpha}^Tk_z$.

$$\begin{array}{l}\left|m\left(z\right)-\hat{m}^\sigma\left(z\right)\right|\\ =\left\|\alpha^Tk_z-\tilde{\alpha}^Tk_z+\tilde{\alpha}^Tk_z-\tilde{\alpha}^T\tilde{k}_z\right\|\end{array} \tag{112}$$

Factoring terms,

$$\begin{array}{l}\left|m\left(z\right)-\hat{m}^\sigma\left(z\right)\right|\\ \leq\left\|\left(\alpha^T-\tilde{\alpha}^T\right)k_z+\tilde{\alpha}^T\left(k_z-\tilde{k}_z\right)\right\|\end{array} \tag{113}$$

The bound on the norm is then evaluated. Using properties c and d.

$$\begin{array}{l}\left|m\left(z\right)-\hat{m}^\sigma\left(z\right)\right|\\ \leq\left\|\left(\alpha^T-\tilde{\alpha}^T\right)\right\|\left\|k_z\right\|+\left\|\tilde{\alpha}^T\right\|\left\|\left(k_z-\tilde{k}_z\right)\right\|\end{array} \tag{114}$$

Using property f.

$$\begin{array}{l}\left|m\left(z\right)-\hat{m}^\sigma\left(z\right)\right|\leq\\ \left\|\left(\alpha-\tilde{\alpha}\right)\right\|\left\|k_z\right\|+\left\|\tilde{\alpha}\right\|\left\|\left(k_z-\tilde{k}_z\right)\right\|\end{array} \tag{115}$$

We now have an expression for the desired bound in terms of several known quantities, which we will proceed to compute the bounds of. We start by reviewing the definitions of the parameters $\tilde{\alpha}$, and α.

$$\tilde{\alpha}=\left(\tau^{-2}\tilde{K}+\omega^2I\right)^{-1}y \tag{116}$$

$$\alpha=\left(\tau^{-2}K+\omega^2I\right)^{-1}y \tag{117}$$

Their difference is written as follows.

$$\begin{array}{l}\tilde{\alpha}-\alpha\\ =\left(\tau^{-2}\tilde{K}+\omega^2I\right)^{-1}y-\left(\tau^{-2}K+\omega^2I\right)^{-1}y\end{array} \tag{118}$$

Factoring,

$$\begin{array}{l}\tilde{\alpha}-\alpha\\ =\left[\left(\tau^{-2}\tilde{K}+\omega^2I\right)^{-1}-\left(\tau^{-2}K+\omega^2I\right)^{-1}\right]\cdot y\end{array} \tag{119}$$

We now use property a,

$$A^{-1}+B^{-1}=B^{-1}\left(A+B\right)A^{-1} \tag{120}$$

A and B are chosen as follows.

$$A=-\left(\tau^{-2}K+\omega^2I\right)^{-1} \tag{121}$$

$$B=\left(\tau^{-2}\tilde{K}+\omega^2I\right)^{-1} \tag{122}$$

The expression for the difference is then written in terms of a product of terms.

$$\begin{array}{l}\tilde{\alpha}-\alpha=\\ \left(\tau^{-2}\tilde{K}+\omega^2I\right)^{-1}\tau^{-2}\left(\tilde{K}-K\right)\left(\tau^{-2}K+\omega^2I\right)^{-1}y\end{array} \tag{123}$$

Now we compute the norm.

$$\begin{aligned}&\|\tilde{\alpha}-\alpha\| \\ &= \left\|\left(\tau^{-2}\tilde{K}+\omega^2 I\right)^{-1}\tau^{-2}\left(\tilde{K}-K\right)\left(\tau^{-2}K+\omega^2 I\right)^{-1}y\right\|\end{aligned} \tag{124}$$

Using property e,

$$\|\tilde{\alpha}-\alpha\| \leq \frac{\left\|\left(\tau^{-2}\tilde{K}-\tau^{-2}K\right)\right\|\|y\|}{\left\|\left(\tau^{-2}\tilde{K}+\omega^2 I\right)\left(\tau^{-2}K+\omega^2 I\right)\right\|} \tag{125}$$

Factoring the constant τ out of the norm.

$$\|\tilde{\alpha}-\alpha\| \leq \frac{\left\|\left(\tilde{K}-K\right)\right\|\|y\|}{\tau^2\left\|\left(\frac{\tilde{K}}{\tau^2}+\omega^2 I\right)\right\|\left\|\left(\frac{K}{\tau^2}+\omega^2 I\right)\right\|} \tag{126}$$

Using property h.

$$\|\tilde{\alpha}-\alpha\| \leq \frac{\left\|\left(\tilde{K}-K\right)\right\|\|y\|}{\tau^2\lambda_{\min}\left(\frac{\tilde{K}}{\tau^2}+\omega^2 I\right)\lambda_{\min}\left(\frac{K}{\tau^2}+\omega^2 I\right)} \tag{127}$$

We now use property h.

$$\lambda_{\min}\left(A+B\right) \geq \lambda_{\min}\left(A\right)+\lambda_{\min}\left(B\right) \tag{128}$$

Because A and B are positive definite, so are their eigenvalues.

$$A>0, B>0 \rightarrow \lambda_{\min}\left(A\right)>0, \lambda_{\min}\left(B\right)>0 \tag{129}$$

Therefore, we transform property h to one that is more useful here.

$$\lambda_{\min}\left(A+B\right) \geq \lambda_{\min}\left(A\right) \tag{130}$$

We now examine the bound of the difference of $\tilde{\alpha}-\alpha$ and simplify utilizing the above property.

$$\|\tilde{\alpha}-\alpha\| \leq \frac{\left\|\left(\tilde{K}-K\right)\right\|\|y\|}{\tau^2\lambda_{\min}\left(\omega^2 I\right)\lambda_{\min}\left(\omega^2 I\right)} \tag{131}$$

Since $\omega^2 I$ is a diagonal matrix of ω^2, its eigenvalues are ω^2.

$$\|\tilde{\alpha}-\alpha\| \leq \frac{\left\|\left(\tilde{K}-K\right)\right\|\|y\|}{\tau^2\omega^4} \tag{132}$$

We now bound the kernel, k_z by the maximum kernel κ normalized by the time τ.

$$\|k_z\| \leq \frac{\kappa}{\sqrt{\tau}} \tag{133}$$

We bound the observation y by that deduced from the infinity bound given in the theorem.

$$\|y\| \leq \sqrt{\tau}M^{\sigma} \tag{134}$$

The bound $\left\|(\alpha-\tilde{\alpha})\right\|\|k_z\|$ is the product of that for $\left\|(\alpha-\tilde{\alpha})\right\|$ and $\|k_z\|$ previously deduced.

$$\left\|(\alpha-\tilde{\alpha})\right\|\|k_z\| \leq \left(\frac{\left\|\left(\tilde{K}-K\right)\right\|\|y\|}{\tau^2\omega^4}\right)\left(\frac{\kappa}{\sqrt{\tau}}\right) \tag{135}$$

Factoring the bound $\|y\|$,

$$\begin{aligned}&\left\|(\alpha-\tilde{\alpha})\right\|\|k_z\| \leq \\ &\left(\frac{\left\|\left(\tilde{K}-K\right)\right\|}{\tau^2\omega^4}\right)\left(\sqrt{\tau}M^{\sigma}\right)\left(\frac{k}{\sqrt{\tau}}\right)\end{aligned} \tag{136}$$

We then simplify,

$$\left\|(\alpha-\tilde{\alpha})\right\|\|k_z\| \leq \left(\frac{\kappa M^{\sigma}\left\|\left(\tilde{K}-K\right)\right\|}{\tau^2\omega^4}\right) \tag{137}$$

This expression involves the bound $\left\|\left(\tilde{K}-K\right)\right\|$ which we will now evaluate.

$$\left\|\tilde{K}-K\right\|^2 = \sum_{i=1}^{\tau}\sum_{j=1}^{\tau}\left(k\left(z_i,z_j\right)-k\left(c_{\upsilon_i},c_{\upsilon_j}\right)\right)^2 \tag{138}$$

We transform the difference of these kernels to an inner product in the basis space.

$$\begin{aligned}&\left\|\tilde{K}-K\right\|^2 = \\ &\sum_{i=1}^{\tau}\sum_{j=1}^{\tau}\left(2\left\langle\psi\left(z_i\right)-\psi\left(c_{\upsilon_i}\right),\psi\left(c_{\upsilon_j}\right)\right\rangle\right)^2\end{aligned} \tag{139}$$

We now factor out the constant term.

$$\begin{aligned}&\left\|\tilde{K}-K\right\|^2 = \\ &4\sum_{i=1}^{\tau}\sum_{j=1}^{\tau}\left\langle\psi\left(z_i\right)-\psi\left(c_{\upsilon_i}\right),\psi\left(c_{\upsilon_j}\right)\right\rangle^2\end{aligned} \tag{140}$$

We now transform the squared inner product to a sum of squared norms and use property c.

$$\begin{aligned}&\left\|\tilde{K}-K\right\|^2 \leq \\ &4\sum_{i=1}^{\tau}\sum_{j=1}^{\tau}\left\|\psi\left(z_i\right)-\psi\left(c_{\upsilon_i}\right)\right\|^2\left\|\psi\left(c_{\upsilon_j}\right)\right\|^2\end{aligned} \tag{141}$$

We now bound the sum by the maximum values of the norms within.

$$\begin{aligned}&\left\|\tilde{K}-K\right\|^2 \leq \\ &4\tau^2\max_i\left\|\psi\left(z_i\right)-\psi\left(c_{\upsilon_i}\right)\right\|^2\left\|\psi\left(c_{\upsilon_j}\right)\right\|^2\end{aligned} \tag{142}$$

Each basis function is bounded by κ, so their square is bounded by κ^2

$$\left\|\psi\left(c_{\upsilon_j}\right)\right\|^2 \leq \kappa \tag{143}$$

As given in the squared norm of the difference of basis functions $\psi(z_i)$ and $\psi(c_{\upsilon_i})$ is k_max.

$$\left\|\tilde{K}-K\right\|^2 \leq 4\tau^2\kappa^2 k_{\max} \tag{144}$$

We now rewrite the bound $\tilde{K}-K$ to incorporate these.

$$\left\|\tilde{K}-K\right\| \leq 2\tau\kappa\sqrt{k_{\max}} \tag{145}$$

The bound on $\alpha-\tilde{\alpha}$ is then written to incorporate the updated bound on $\tilde{K}-K$.

$$\|\alpha-\tilde{\alpha}\|\,\|k_z\| \leq \frac{\kappa M^\sigma \left\|\tilde{K}-K\right\|}{\tau\omega^4} \tag{146}$$

Plugging in the bound for $\tilde{K}-K$,

$$\|\alpha-\tilde{\alpha}\|\,\|k_z\| \leq \frac{\kappa M^\sigma \left(2\tau\kappa\sqrt{k_{\max}}\right)}{\tau\omega^4} \tag{147}$$

Simplifying,

$$\|\alpha-\tilde{\alpha}\|\,\|k_z\| \leq \frac{2\kappa^2 M^\sigma \sqrt{k_{\max}}}{\omega^4} \tag{148}$$

We now have the desired bound on the first term in the bound for ϵ. For the second, desire a bound on $\tilde{\alpha}$. We start with the expression for $\tilde{\alpha}$.

$$\|\tilde{\alpha}\| \leq \left\|\left(\frac{1}{\tau^2}\tilde{K}+\omega^2 I\right)^{-1}\right\|\,\|y\| \tag{149}$$

Using property e,

$$\|\tilde{\alpha}\| \leq \frac{\|y\|}{\left\|\left(\frac{1}{\tau^2}\tilde{K}+\omega^2 I\right)\right\|} \tag{150}$$

Using property g.

$$\|\tilde{\alpha}\| \leq \frac{\|y\|}{\lambda_{\min}\left(\frac{1}{\tau^2}\tilde{K}+\omega^2 I\right)} \tag{151}$$

We use property h and simplify as before.

$$\|\tilde{\alpha}\| \leq \frac{\|y\|}{\lambda_{\min}(\omega^2)} \tag{152}$$

Observing the minimum eigenvalues of $I\omega^2$ is ω^2.

$$\|\tilde{\alpha}\| \leq \frac{\sqrt{\tau}M^\sigma}{\omega^2} \tag{153}$$

Now, we compute the bound on $\left\|k_z-\tilde{k}_z\right\|$. Writing this in terms of a normalized vector of kernel differences,

$$\left\|k_z-\tilde{k}_z\right\| = \frac{1}{\tau}\left\|\begin{pmatrix} k(z_1,z)-k(c_{\upsilon_1},z) \\ \vdots \\ k(z_\tau,z)-k(c_{\upsilon_\tau},z)\end{pmatrix}\right\| \tag{154}$$

We now write this vector as a sum of inner products in the kernel space.

$$\left\|k_z-\tilde{k}_z\right\| = \frac{1}{\tau}\sum_{i=1}^{\tau}\left\|\left\langle \psi(z_i)-\psi(c_{\upsilon_i}),\psi(z)\right\rangle\right\| \tag{155}$$

The sum is bounded by τ times its maximum value.

$$\left\|k_z-\tilde{k}_z\right\| \leq \frac{1}{\tau}\left\|\tau \max_i \left\langle \psi(z_i)-\psi(c_{\upsilon_i}),\psi(z)\right\rangle\right\| \tag{156}$$

Factoring our constants,

$$\begin{aligned}&\left\|k_z-\tilde{k}_z\right\| \leq \\ &\tfrac{\sqrt{\tau}}{\tau}\max_i \left\|\psi(z)\right\| \max_i \left\|\psi(z_i)-\psi(c_{\upsilon_i})\right\|\end{aligned} \tag{157}$$

Bounding the norms by κ and k_{max},

$$\left\|k_z-\tilde{k}_z\right\| \leq \frac{\kappa k_{\max}}{\sqrt{\tau}} \tag{158}$$

We now write the second term in the bound for ϵ as a product of the bounds.

$$\|\tilde{\alpha}\|\left\|k_z-\tilde{k}_z\right\| \leq \left(\frac{\sqrt{\tau}M^\sigma}{\omega^2}\right)\left(\frac{k k_{\max}}{\sqrt{\tau}}\right) \tag{159}$$

Simplifying yields the desired expression.

$$\|\tilde{\alpha}\|\left\|k_z-\tilde{k}_z\right\| \leq \frac{k k_{\max} M^\sigma}{\omega^2} \tag{160}$$

We now write the bound for ϵ.

$$\left|m(z)-\hat{m}^\sigma(z)\right| \leq \left\{\begin{array}{c} \|\alpha-\tilde{\alpha}\|\,\|k_z\| \\ +\|\tilde{\alpha}\|\left\|k_z-\tilde{k}_z\right\| \end{array}\right\} \tag{161}$$

Plugging in the bounds for these terms yields the desired expression.

$$\left|m(z)-\hat{m}^\sigma(z)\right| \leq \frac{2k^2 M^\sigma\sqrt{k_{\max}}}{\omega^4} + \frac{k k_{\max} M^\sigma}{\omega^2} \tag{162}$$

■

Theorem 5: Consider the following system.

$$\left\{\begin{array}{c} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = f\left(x(t)\right) + b\left(x(t)\right)\delta(t) \end{array}\right\} \tag{163}$$

The control law is.

$$\delta = \hat{b}^{-1}(x)\left(\upsilon - \hat{f}(x)\right) \tag{164}$$

The psuedoinput is:

$$\upsilon = \upsilon_{rm} + \upsilon_{pd} - \upsilon_{ad} \tag{165}$$

Finally, the model uncertainty is given by a Gaussian process.

$$\Delta \sim GP(m,k) \tag{166}$$

If the adaptive psuedoinput is given as:

$$\upsilon_{ad} = \hat{m}^\sigma(z) \tag{167}$$

Where this estimate is the result of a Gaussian process estimator as describe above. Then, we guarantee that the system is mean square uniformly ultimately bounded a.s. and proportion to the tolerance ε_{tol}.

Proof: Define the following stochastic candidate Lyupanov function:

$$V\left(e\left(t\right)\right)=\frac{1}{2}e^{T}\left(t\right)Pe\left(t\right),P>0 \quad (168)$$

We deduce the standard bound based on the eigenvalues of P.

$$\frac{1}{2}\lambda_{\min}\left(P\right)\left\|e\right\|^{2}\leq V\left(e\right)\leq\frac{1}{2}\lambda_{\max}\left(P\right)\left\|e\right\|^{2} \quad (169)$$

We then write the differential generator in terms of the Lyupanov function as before.

$$\begin{array}{l}\frac{d\left[V(e)\right]}{dt}=\\ \frac{\partial\left[V(e)\right]}{\partial t}+\sum_{j}\left[F_{j}\left(t,x\right)+H_{\sigma,ij}\left(t,x\right)d\xi\right]\frac{\partial\left[V(t,x)\right]}{\partial e_{j}}\\ +\frac{1}{2}\sum_{i,j}\left[H_{\sigma}H_{\sigma}^{T}\right]_{ij}\left(t,x\right)\frac{\partial^{2}\left[V(e)\right]}{\partial e_{j}\partial e_{i}}\end{array} \quad (170)$$

Recall the stochastic differential equation for the error.

$$de=Aedt+B\left[\varepsilon_{m}^{\sigma}\left(z\right)dt+G_{\sigma}d\xi\right] \quad (171)$$

We can read off the parameters F and H_{σ}.

$$F\left(t,x\right)=Ae+B\varepsilon_{m}^{\sigma}\left(z\right),H_{\sigma}\left(t,x\right)=BG_{\sigma} \quad (172)$$

The differential generator is then written for the error equation as follows.

$$\begin{array}{l}\frac{dE\left[V(e)\right]}{dt}=\sum_{j}\frac{\partial E\left[V(e)\right]}{\partial e_{j}}\left[Ae_{j}+B\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right]\\ +\frac{1}{2}\sum_{i,j}\left[BG_{\sigma}(BG_{\sigma})^{T}\right]_{ij}\frac{\partial^{2}E\left[V(e)\right]}{\partial e_{j}\partial e_{i}}\end{array} \quad (173)$$

We first consider the first term and note that the sum can be removed.

$$\sum_{j}\frac{\partial\left[V\left(e\right)\right]}{\partial e_{j}}=\frac{\partial V\left(e\right)}{\partial e} \quad (174)$$

Then, it can be shown that this first term is just the time derivative of the Lyupanov function $V(t,x)$.

$$\frac{\partial V\left(e\right)}{\partial e}\left[Ae+B\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right]=\frac{\partial V\left(e\right)}{\partial e}\frac{\partial e}{\partial t} \quad (175)$$

$$\frac{\partial V\left(e\right)}{\partial e}\left[Ae+B\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right]=\frac{\partial V\left(e\right)}{\partial t} \quad (176)$$

This is a known result from linear algebra.

$$\begin{array}{l}\frac{\partial V(e)}{\partial t}=\frac{1}{2}\left(Ae+B\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right)^{T}Pe\\ +\frac{1}{2}+e^{T}P\left(Ae+B\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right)\end{array} \quad (177)$$

The terms involving A and B are broken up.

$$\begin{array}{l}\frac{\partial V(e)}{\partial t}=\frac{1}{2}(Ae)^{T}Pe+e^{T}P\left(Ae\right)\\ +\frac{1}{2}\left(B^{T}Pe+e^{T}PB\right)\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\end{array} \quad (178)$$

Next the transposes are evaluated.

$$\begin{array}{l}\frac{\partial V(e)}{\partial t}=\frac{1}{2}e^{T}A^{T}Pe+e^{T}PAe\\ +\frac{1}{2}\left(B^{T}Pe+e^{T}PB\right)\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\end{array} \quad (179)$$

Finally, the expression is factored.

$$\begin{array}{l}\frac{\partial V(e)}{\partial t}=\frac{1}{2}e^{T}\left(A^{T}P+PA\right)e\\ +\frac{1}{2}\left(\left(e^{T}PB\right)^{T}+e^{T}PB\right)\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\end{array} \quad (180)$$

Now we use the Lyupanov equation and the fact that the second set of terms is scalar.

$$-Q=A^{T}P+PA,\left(e^{T}PB\right)^{T}=e^{T}PB \quad (181)$$

The first terms will simplify to that shown below.

$$\frac{\partial V\left(e\right)}{\partial t}=\left[-\frac{1}{2}e^{T}Qe+e^{T}PB\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right] \quad (182)$$

Now the second term can be written trivially in terms of the trace.

$$\begin{array}{l}\frac{1}{2}\sum_{i,j}\left[BG_{\sigma}(BG_{\sigma})^{T}\right]_{ij}\frac{\partial^{2}\left[V(e)\right]}{\partial e_{j}\partial e_{i}}\\ =\frac{1}{2}Tr\left(BG_{\sigma}(BG_{\sigma})^{T}P\right)\end{array} \quad (183)$$

The final equation for the differential generator is below.

$$\begin{array}{l}\frac{dV(e)}{dt}=\left[-\frac{1}{2}e^{T}Qe+e^{T}PB\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right]\\ +\frac{1}{2}Tr\left(BG_{\sigma}(BG_{\sigma})^{T}P\right)\end{array} \quad (184)$$

We now desire a bound on the differential generator. We first use the previous bound involving the eigenvalues.

$$\frac{1}{2}\lambda_{\min}\left(P\right)\left\|e\right\|^{2}\leq V\left(e\right)\leq\frac{1}{2}\lambda_{\max}\left(P\right)\left\|e\right\|^{2} \quad (185)$$

We examine the equation for the differential generator and take the norm of the whole equation.

$$\begin{array}{l}\frac{dV(e)}{dt}=\left[-\frac{1}{2}e^{T}Qe+e^{T}PB\left(\varepsilon_{m}^{\sigma}+G_{\sigma}d\xi\right)\right]\\ +\frac{1}{2}Tr\left(BG_{\sigma}(BG_{\sigma})^{T}P\right)\end{array} \quad (186)$$

$$\begin{array}{l}\frac{dV(e)}{dt}\leq-\frac{1}{2}\lambda_{\min}\left(Q\right)\cdot\left\|e\right\|^{2}+\frac{1}{2}\left\|P\right\|\cdot\left\|BG_{\sigma}\right\|^{2}\\ +\left\|PB\right\|\cdot\left\|e\right\|\cdot\left(\left\|\varepsilon_{m}^{\sigma}\right\|+\left\|G_{\sigma}d\xi\right\|\right)\end{array} \quad (187)$$

We define the following constants.

$$c_{1}=\frac{1}{2}\left\|P\right\|\cdot\left\|BG_{\sigma}\right\|^{2},c_{2}=\left\|PB\right\|,c'=\left\|G_{\sigma}d\xi\right\| \quad (188)$$

The norm then simplifies as follows.

$$\begin{array}{l}\frac{dV(e)}{dt}\leq\\ -\frac{1}{2}\lambda_{\min}\left(Q\right)\cdot\left\|e\right\|^{2}+c_{2}\cdot\left\|e\right\|\cdot\left(\left\|\varepsilon_{m}^{\sigma}\right\|+c'\right)+c_{1}\end{array} \quad (189)$$

From Theorem 4, we have the following bound on ϵ.

$$\left\|\varepsilon_{m}^{\sigma}\right\|\leq c_{3}M^{\sigma}\sqrt{k_{\max}^{\sigma}}+c_{4}M^{\sigma}k_{\max}^{\sigma} \quad (190)$$

The normed equation is further simplified.

$$\begin{array}{l}\frac{dV(e)}{dt}\leq-\frac{1}{2}\lambda_{\min}\left(Q\right)\cdot\left\|e\right\|^{2}+c_{1}\\ +c_{2}\cdot\left\|e\right\|\cdot\left(c_{3}M^{\sigma}\sqrt{k_{\max^{\sigma}}}+c_{4}M^{\sigma}k_{\max^{\sigma}}+c'\right)\end{array} \quad (191)$$

We now define one further constant to simplify the equation

$$c_{5}^{\sigma}M^{\sigma}=c_{2}\left(c_{3}M^{\sigma}\sqrt{k_{\max}^{\sigma}}+c_{4}M^{\sigma}k_{\max}^{\sigma}+c'\right) \quad (192)$$

The final equation is now written as a quadratic in the error, which must be less than or equal to zero.

$$-\frac{1}{2}\lambda_{\min}\left(Q\right)\cdot\left\|e\right\|^{2}+\left(c_{5}^{g}M^{\sigma}\right)\cdot\left\|e\right\|+c_{1}\leq 0 \tag{193}$$

The solution to this quadratic bounds the error,

$$\left\|e\right\|\geq\frac{c_{5}^{g}M^{\sigma}+\sqrt{\left(c_{5}^{g}M^{\sigma}\right)^{2}+2\lambda_{\min}Qc_{1}}}{\lambda_{\min}Q} \tag{194}$$

The resultant bound on the differential generator bounds stability as desired.

$$\frac{dV\left(e\right)}{dt}\overset{a.s.}{\leq}0 \tag{195}$$

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V. Trajectory Tracking in Presence of Wing Rock Dynamics in an Unknown Operating Domain

In this section, the paper validated the Gaussian Process Model Reference Adaptive Control architecture for the Wing Rock Problem, a nonlinear control problem encountered during landing of aircraft [5]. The model was compared against a fixed center Radial Basis Function approach and the performance improvement was observed.

A. System Within Domain of Operation

This system performed more effectively than the Radial Basis Function approach even within the domain of operation, showing less oscillations while maintaining low error and high confidence.

B. System Driven Outside Domain of Operation

The two controllers were again compared when the Wing Rock System is driven outside the domain of operation. Here, the Gaussian Process MRAC architecture is significantly better than the RBF approach, demonstrating that this architecture is much better at mitigating oscillations associated with model uncertainty.

VI. Conclusion

The Gaussian Process Model Reference Adaptive Control Architecture which was demonstrated here provides a more robust control framework for dynamical systems with a significant amount of model uncertainty. This approach improves over past methods, which relied on pre-allocated information about the domain of operation, and were not robust to changes in operating conditions. After introducing the architecture used here, this paper provides a theoretical framework for bounding the error in the model and proving system stability. Experiments were then run which successfully demonstrated the improvement of this architecture over previous methods.

References

[1] G. Chowdhary, H. A. Kingravi, J. P. How and P. A. Vela, "Bayesian Nonparametric Adaptive Control Using Gaussian Processes," in IEEE Transactions on Neural Networks and Learning Systems, vol. 26, no. 3, pp. 537-550, March 2015.

[2] Csat and M. Opper, "Sparse On-Line Gaussian Processes," in Neural Computation, vol. 14, no. 3, pp. 641-668, March 1 2002.

[3] Y. Miyahara, Ultimate boundedness of the systems governed by stochastic differential equations, Nagoya Math J., vol. 47, no. 1, pp. 111144, 1972.

[4] R. Dudley, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, J. Funct. Anal., vol. 1, no. 3, pp. 290330, 1967.

[5] A. A. Saad, Simulation and analysis of wing rock physics for a generic fighter model with three degrees of freedom, Ph.D. dissertation, Air Force Inst. Technol., Air Univ., Wright-Patterson Air Force Base, Dayton, OH, USA, 2000.