On topological obstructions to global stabilization of an inverted pendulum [1]

Wyatt McAllister[1]

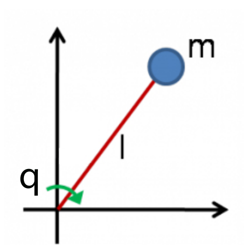
Abstract

This paper considers the classical problem of an inverted pendulum, with the additional spatial constraint that the pendulum cannot leave the upper half-plane of its state space (i.e. cannot go below the horizontal plane). Under this assumption, it is shown that there exists no continuous control which can be applied to achieve global asymptotic stability. Similar results are shown to hold for the analogous systems of a pendulum with torque control, with viscous friction, and for systems such as the spherical pendulum and cart-pole.

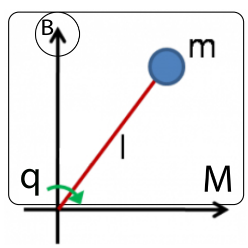
I. Introduction

In many physical systems, there exists a spatial constraint which limits the reachable state space. In the case of a pendulum, the pivot could be placed on a horizontal surface which prevents the pendulum from falling below the horizontal position. This paper sets out to answer the question of whether this spatial constraint causes global asymptotic stabilization to be impossible for this system under any continuous control. It is shown that this is not only true for a variety of systems, but that it is true even when the equilibrium is an attractor for its closed neighborhood.

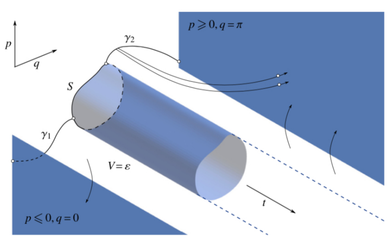
II. Simple Inverted Pendulum



(a) Simple Inverted Pendulum



(b) Visualization of M and B



(c) Exit Sets for the State Space

Fig. 1: Theorem II.1

Let q be the position and p be the angular momentum. Let u be a force control.

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= u\left(q, p, t\right)\sin q - \cos q \qquad u \in \mathrm{Lip}\left(\mathbb{R}^3, \mathbb{R}\right) \\ m &= g = l = 1 \end{aligned} \tag{1}$$

Theorem II.1. ***Theorem 2.1:***

- *Let $u\left(q, p, t\right) \in Lip\left(\mathbb{R}^3, \mathbb{R}\right)$ be a force control for (1).*
- *Let $M := \{(q, p) : 0 < q < \pi\}$.*
- *Let $\partial M := \{(q, p) : p \neq 0\}$.*
- *Let $\mu \in M \backslash \partial M$, be a locally asymptotically stable equilibrium for (1).*
- *Let $B := \left\{x := (q, p) \approx \mu\right\} \subset M$ be a closed neighborhood of μ.*

Then, if there exists a Lyapunov function V satisfying conditions $(L1)$ and $(L2)$ as below, there exists an initial condition and a neighborhood of the equilibrium such that all solutions which start in $M \backslash B$ will stay there for all time.

$$\exists V \quad \begin{array}{l} L1 : V\left(\mu\right) = 0, V > 0 \quad \forall x \in B \backslash \mu \\ L2 : \dot{V} < 0 \quad \forall x \in B \backslash \mu \quad \forall t \in \mathbb{R}^+ := \{t > t_0\} \end{array} \Rightarrow \exists \left(q_0, p_0\right), B : \left(q\left(t, q_0, p_0\right), p\left(t, q_0, p_0\right)\right) \in M \backslash B \quad \forall t \in \mathbb{R}^+$$

[1] Wyatt McAllister is with the Dept. of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign

The sets M and B can be visualized as in 1b.

Proof. **Show:** There are solutions that remain in $M \backslash B$.

Assume: All solutions leave $M \backslash B$.

(i) Construct a mapping from $M \backslash B$ to the exit set.
(ii) Show that this mapping is continuous.
(iii) Show that no solution can leave $M \backslash B$ through a point with zero derivatives.
(iv) The "continuous" mapping is from connected to multiple distinct disconnected sets: Contradiction!

Step 1: Let $S := \{x : V(x) = \varepsilon > 0, \varepsilon \in \mathbb{R}\}$ be a level set of the Lyapunov function V.

Let γ_1 and γ_2 be curves in the solution space which connect S with the exit sets $\{q = 0, p < 0\}$ and $\{q = \pi, p > 0\}$, respectively.

$$\gamma_1 : S \to \{q = 0, p < 0\}$$
$$\gamma_2 : S \to \{q = \pi, p > 0\}$$
$$\gamma_1 \cap \gamma_2 = \emptyset$$

Then, σ maps $(S \cup \gamma_1 \cup \gamma_2)$ at $t = t_0$ to the exit set.

$$\sigma : (S \cup \gamma_1 \cup \gamma_2) \times \{t_0\} \to \left(S, \{q = 0, p < 0\}, \{q = \pi, p > 0\}\right) \times \mathbb{R}^+$$

Step 2: System (1) is Lipschitz, so all solutions depend continuously on initial conditions. Therefore, it suffices to examine initial conditions on the boundaries of $M \backslash B$ and show they evolve outside $M \backslash B$.

Let W be the set starting on the boundaries of $M \backslash B$.

$$W := \left(S, \left(\gamma_1 \cap \{q = 0, p < 0\}\right), \left(\gamma_2 \cap \{q = \pi, p > 0\}\right)\right) \times \{t_0\}$$

For any initial condition in W, solutions will evolve outside $M \backslash B$ within δ of t_0. This comes from the system equations and the definition of S.

$$\forall (q_0, p_0, t_0) \in W, \exists \delta : \left(q(t_0 + \delta, q_0, p_0), p(t_0 + \delta, q_0, p_0)\right) \notin M \backslash B \times \mathbb{R}^+$$

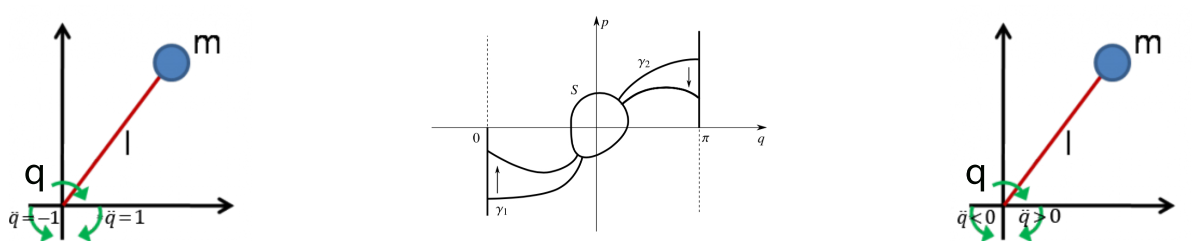
For this set, σ is the identity, and thus is continuous.

$$\sigma(q_0, p_0, t_0) = (q_0, p_0, t_0) \quad \forall W$$

Step 3: Consider the states in which the angular momentum p is zero on the boundary.

$$(q, p) = (0, 0) \Rightarrow \dot{q} = p = 0, \ddot{q} = \dot{p} = u(q, p, t)\sin q - \cos q = -1$$
$$(q, p) = (\pi, 0) \Rightarrow \dot{q} = p = 0, \ddot{q} = \dot{p} = u(q, p, t)\sin q - \cos q = 1$$

These points are already outside of $M \backslash B$.



(a) Visualization of Derivatives (TII.1) (b) Exit Set of Solution Paths (RII.2) (c) Visualizing Derivatives for (CII.3)

Fig. 2: Theorem (T) II.1, Remark (R) II.2, and Corollary (C) II.3

All solutions in γ_1 or γ_2 cannot leave $M \backslash B$ through points where $p = 0$.

Step 4: Since no solution can leave $M \backslash B$ through a point with zero derivatives, we must map our original starting set, which was connected, to a disconnected set.

$$(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\} \to (S, \gamma_1 \cap \partial M, \gamma_2 \cap \partial M) \times \{t_0\}$$

This set is disconnected because γ_1 and γ_2 do not intersect.

$$\gamma_1 \cap \gamma_2 = \emptyset \Rightarrow (\gamma_1 \cap \partial M) \cap (\gamma_2 \cap \partial M) = \emptyset$$

It is also because, by definition, S is an attractor for the equilibrium outside ∂M. Therefore, ∂M separates the set S from γ_1 and γ_2.

$$S \times \mathbb{R}^+ = B := \{x \approx \mu\} \subset M \backslash \partial M \Rightarrow \left(S \times \mathbb{R}^+\right) \cap \partial M = \emptyset$$

Thus, we see that the following two maps are equivalent.

$$\begin{aligned}
&\sigma : (S \cup \gamma_1 \cup \gamma_2) \times \{t_0\} \to (S, \gamma_1 \cap \partial M, \gamma_2 \cap \partial M) \times \{t_0\} \\
&\Leftrightarrow \\
&\sigma : (S \cup \gamma_1 \cup \gamma_2) \times \{t_0\} \to \left(S, \{q = 0, p < 0\}, \{q = \pi, p > 0\}\right) \times \mathbb{R}^+
\end{aligned}$$

Since this is impossible, we arrive at a contradiction, and therefore know that there are solutions starting in $M \backslash B$ which remain there for all time. The proof is then complete. ■

Remark II.2. *We have shown that, for our choice of γ_1 and γ_2, there are solutions which remain in $M \backslash B$.*

In fact, by varying γ_1 and γ_2, we can construct a whole family of solution paths which remain in $M \backslash B$ for all time. Since none of these solutions approach the equilibrium for any control, this equilibrium cannot be a global attractor. Therefore, Theorem II.1 shows that there does not exist any continuous control which will globally stabilize (1).

Corollary II.3. *Consider the following inverted pendulum system. Let q be the position and p be the angular momentum. Let u be a force control and w be a torque control.*

$$\begin{aligned}
\dot{q} &= p \\
\dot{p} &= u\,(q, p, t)\sin q - \cos q + w\,(q, p, t)
\end{aligned} \tag{2}$$

Here, u and w are defined as before, and m, g, l are unity.

$$u, w \in \mathrm{Lip}\left(R^3, R\right) \quad m, g, l = 1$$

Theorem 2.1 still holds for this system.

Proof. Solutions are still outside $M \backslash B$ at $(p, q) = (0, 0)$ and $(p, q) = (\pi, 0)$ under the following conditions on w.

$$\begin{aligned}
&w\,(0, 0, t) < 1 \Rightarrow \ddot{q} = \dot{p} < 0 \\
&w\,(\pi, 0, t) > -1 \Rightarrow \ddot{q} = \dot{p} > 0
\end{aligned}$$

The rest of the proof remains the same. ■

Now, we will examine the pendulum system where viscous friction is added.

As before, we will let q be the position and p be the angular momentum. Now, ν is the coefficient of viscous friction. Let u be a force control.

$$\begin{aligned}
&\dot{q} = p \\
&\dot{p} = u\,(q, p, t)\sin q - \cos q - \nu p, \quad u \in \mathrm{Lip}\left(\mathbb{R}^3, \mathbb{R}\right) \\
&m = g = l = 1
\end{aligned} \tag{3}$$

Consider the following definitions and lemma.

Definition II.4. *Let $W \subset \mathbb{R}^3$ be a T - periodic segment for (2) if $W = W_0 \times [0, T]$, where initial state $W_0 \subset \mathbb{R}^2$ is real and compact.*

Definition II.5. *For all points $(t_0, q_0, p_0) \in W$, we say they are in the exit set W^- if the following holds.*

$$\forall t \in [0, \delta], \exists \delta > 0 : \left(q\,(t_0 + t, q_0, p_0), p\,(t_0 + t, q_0, p_0), t_0 + t\right) \notin W$$

Definition II.6. *We say W is a simple T - periodic segment if its exit set is the union of its terminal state with the exit set for the initial state W_0.*

$$W^- = W_0^- \times [0, T] \cup (W \cap \{t = T\}), W_0^- \subset \mathbb{R}^2$$

Definition II.7. *For a simple T - periodic segment W, we denote the essential exit set W^{--} as below.*

$$W^{--} = W_0^- \times [0, T]$$

Lemma II.8. *If there exists a simple T - periodic segment W for (2), W^{--} is compact, and $\chi(W) - \chi(W^{--}) \neq 0$, then solutions which start in W_0 will stay there for all time.*

$$\exists \left(q\left(t, q_0, p_0\right), p\left(t, q_0, p_0\right)\right) : \left(q\left(t, q_0, p_0\right), p\left(t, q_0, p_0\right)\right) \in W_0 \backslash \partial W_0 \quad \forall t$$

Here, χ is the Euler-Poincare characteristic, defined to be the number of vertices minus the number of edges plus the number of faces for a polyhedron g.

$$\chi(g) = V - E + F$$

Theorem II.9. *For any bounded T - periodic solution to system (2), there exists an initial condition such that the solution remains in $\{0 < q < \pi\}$ for all time.*

$$\forall\left(u, v > 0\right), \exists\left(q_0, p_0\right) \times \{t_0\} : \left(q\left(t, q_0, p_0\right), p\left(t, q_0, p_0\right)\right) \in \{0 < q < \pi\} \times \mathbb{R}^+$$

Proof. Consider the *simple T - periodic segment W*.

$$W = \left\{q, p, t : q \in [0, \pi], p \in [-\rho, \rho], t \in [0, T], \rho > 0\right\}$$

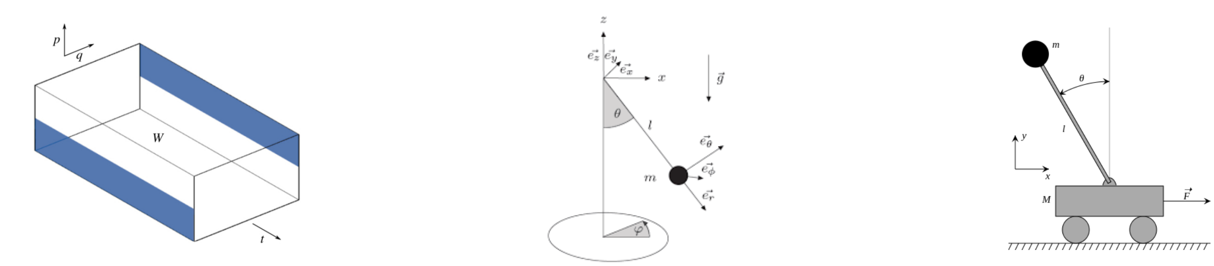
If $q = 0$, $p = \rho$, $q = \pi$, or $p = -\rho$, the solution is already outside the set via the same reasoning as before.

$$\begin{aligned} \dot{p} < 0 \,\forall\,(q, p) : q = 0, &\quad p \in [0, \rho] \\ \dot{p} > 0 \,\forall\,(q, p) : q = \pi, &\quad p \in [-\rho, 0] \end{aligned}$$

We have the essential exit set W^{--}.

$$W^{--} = \left\{q, p, t : q = 0, p \in [-\rho, 0], t \in [0, T]\right\} \cup \left\{q, p, t : q = \pi, p \in [0, \rho], t \in [0, T]\right\}$$

Since $\chi(W) - \chi(W^{--}) \neq 0$ the T2.4 follows from L2.3.



(a) Exit Set Visualization (TII.9)

(b) Spherical Pendulum (TIII.1)

(c) Cart-Pole (TIII.2)

Fig. 3: Theorem (T) II.9, Theorem (T) III.1, Theorem (T) (III.2)

■

III. Systems with two degrees of freedom

Consider the following spherical pendulum system, where φ is the altitude (inclination) angle pf the pendulum and θ is its azimuth angle.

$$\begin{aligned} \ddot{\varphi} + u \sin\varphi \cos\theta + v \sin\varphi \sin\theta + \dot{\theta}^2 \cos\varphi \sin\varphi &= -\cos\varphi \\ \ddot{\theta}\cos^2\varphi - \dot{\theta}\dot{\varphi} \sin\varphi \cos\varphi + u \sin\theta \cos\varphi - v \cos\varphi \cos\theta &= 0 \end{aligned} \tag{4}$$

Let (u, v) be Lipschitz acceleration controls of the pivot point (along the projections of the two axis into the horizontal plane). We take the mass, gravitational constant, and length to be unity as before. Let $(u, v) \in \mathrm{Lip}\left(\mathbb{R}^5, \mathbb{R}\right)$ and $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Consider the equations,

$$u = u\left(\varphi, \dot{\varphi}, \theta, \dot{\theta}, t\right), \quad v = v\left(\varphi, \dot{\varphi}, \theta, \dot{\theta}, t\right)$$

subject to $m = g = l = 1$. Also as before, let M be the upper half-plane, ∂M be the set of states with nonzero derivatives, and $\mu \in M \backslash \partial M$ be an equilibrium of the system.

$$\begin{aligned} M &:= \{(\varphi, \theta) : \varphi > 0\} \\ \partial M &:= \left\{(\varphi, \theta) : \dot{\varphi} \neq 0, \dot{\theta} \neq 0\right\} \end{aligned}$$

And lastly, B, and V are defined analogously to before.

$$B := \left\{(\varphi, \theta) \approx \mu\right\}$$
$$V(\varphi, \theta) : (L1), (L2)$$

Theorem III.1. *Let $(u, v) \in Lip(\mathbb{R}^5, \mathbb{R})$ be given control functions, $\mu \in M\backslash B$ be an equilibrium for system (4), and $t_0 \in \mathbb{R}$. Suppose there exists a Lyapunov function V satisfying L1 and L2, then there is an initial condition, and an open neighborhood of the equilibrium, such that all solutions that start in $M\backslash B$ will stay there for all time.*

$$\exists V, \left(\varphi_0, \dot{\varphi}_0, \theta_0, \dot{\theta}_0, t_0\right), B \subset M : \left(\varphi\left(\varphi_0, \dot{\varphi}_0, \theta_0, \dot{\theta}_0, t\right), \theta\left(\varphi_0, \dot{\varphi}_0, \theta_0, \dot{\theta}_0, t\right)\right) \in M\backslash B \quad \forall t \in \mathbb{R}^+$$

Proof. As before, let $S := \{V = \varepsilon > 0, \varepsilon \in \mathbb{R}\}$ be a level set of the Lyapunov function V. However, now we need only consider one curve γ which connects S with the exit set.

$$\gamma : S \to \{\varphi = 0, \dot{\varphi} \leq 0\}$$

Let us assume that all solutions starting in $\gamma \times \{t_0\}$ leave $M\backslash B$. Then, we obtain a map σ.

$$\sigma : (S \cup \gamma) \times \{t_0\} \to \left(\gamma \cap S, \gamma \cap \{\varphi = 0, \dot{\varphi} \leq 0\}\right) \times R^+$$

The rest of the proof proceeds as in Theorem II.1. ■

We now consider the cart-pole system below. Here, q is the angular position of the pendulum, p is the angular momentum, x is the horizontal position of the pivot point, and y is its horizontal velocity. Also, $m > 0$ is the mass of the cart, and $u \in \mathrm{Lip}(\mathbb{R}^5, \mathbb{R})$ is a force control, as before.

$$\begin{aligned}
\dot{q} &= p \\
\dot{p} &= \frac{u(q, p, x, y, t)\sin q + p^2 \sin q \cos q - (1 + m)\cos q}{m + \cos^2 q} \\
\dot{x} &= y \\
\dot{y} &= \left(m + \cos^2 q\right)^{-1}\left(u(q, p, x, y, t) + p^2 \cos q - \sin q \cos q\right)
\end{aligned} \tag{5}$$

Theorem III.2. *Let $u \in Lip(\mathbb{R}^5, \mathbb{R})$ be a given control function, $M := \{(q, p, x, y) : 0 < q < \pi\}$, $\mu \in M\backslash\partial M$ be an equilibrium for system (5). Suppose there exists a Lyapunov function V satisfying L1 and L2, then there exists an initial condition, and an open neighborhood of the equilibrium such that all solutions starting in $M\backslash B$ remain there for all time.*

$$\exists V, (q_0, p_0, x_0, y_0, t_0), B \subset M : \left(q(q_0, p_0, x_0, y_0, t), p(q_0, p_0, x_0, y_0, t)\right) \in M\backslash B \quad \forall t \in \mathbb{R}^+$$

Proof. As in the case of the simple inverted pendulum, we can make the observation that, all points $(q = 0, p = 0)$, and all points $(q = \pi, p = 0))$, are already outside of $M\backslash B$.

$$(q = 0, p = 0) \Rightarrow \ddot{q} < 0$$
$$(q = \pi, p = 0) \Rightarrow \ddot{q} > 0$$

The rest of the proof proceeds the same as Theorem II.1. ■

IV. Conclusions and Critique

In this work, we find that limiting the inverted pendulum to the upper half-plane creates a topological obstruction in the state space. We can show that under this spatial constraint, many solutions never approach the neighborhood of the equilibrium for any Lipschitz control. Thus, there is no globally stabilizing continuous control. This makes a great case for the necessity of hybrid control for the stabilization of these systems.

As a critique of this work, we say that while the paper presents very novel and useful content, it could be presented much more effectively. A portion of this work was published in [3], and the presentation there is much clearer. Not only was this paper poorly written, but the proofs did not contain detailed mathematical explanations. They should have been fleshed out more, and the full proofs for their generalizations should be included in the supplementary material. This author should consult a native speaker to improve this work to the quality of their other submission.

References

[1] Polekhin, I. (2018). On topological obstructions to global stabilization of an inverted pendulum. Systems and Control Letters, 113:3135.

[2] Srzednicki, R. (1994). Periodic and bounded solutions in blocks for time-periodic nonautonomous ordinary differential equations. Nonlinear Analysis: Theory, Methods and Applications, 22(6):707737.

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